Screening with an Approximate Type Space

Madarász and Prat (Working Paper, Dec 2010) summary by N. Antić

Practical mechanism design often employes approximations, e.g. if the true type space is not known. How can we design a mechanism (in a multidimensional screening problem) which is sophisticated enough to take this into account?

Model

- One principal and one agent
- Private-values, quasi-linear utility
- Agent has a private type $t \in T \subset \mathbb{R}^m$
  - $t \sim f \in \Delta T$, with $supp\{f\} = T$
  - T is compact, connected, uncountable and its diameter is denoted $D_T = \sup_{t,t' \in T} \|t - t'\|$
- True type space is a pair (T, f)
- Y is a compact set of alternatives
- Principal offers a menu, $M = \{(y', p'), (y'', p''), ...\}$, where $(y, p) \in Y \times \mathbb{R}$
- Assume there is an outside option in every M, $(y_0, 0) \in M$
- The principal's profit is $\pi(y, p) = p - c(y)$, where $c(y)$ is the cost of producing good y
  - WLOG, assume that $c(y_0) = 0$
  - π is bounded from below by 0
- The agent's utility is $v(t, y, p) = u(t, y) - p$
- Assume that for any y, $u(\cdot, y)$ is Lipschitz continuous, i.e. $\exists K \in \mathbb{R}$ such that $\forall y \in Y$ and $\forall t, t' \in T$,

$$\left| \frac{u(t, y) - u(t', y)}{\|t - t'\|} \right| \leq K$$

  - Need a "smooth" environment to get an approximation
- Assume that

$$0 \leq \pi(y, p) \leq \Pi_{\max} := \sup_{y \in Y, t \in T} u(t, y) - c(y) < \infty$$

- Fix the levels of π and u by assuming that $\Pi_{\max} = 1$ and $D_T = 1$
  - Affine transformations of payoff functions lead to the same problem, so we can normalize in this way
- A class of normalized problems can be described by the Lipschitz constant, K
  - A single screening problem can be described by (T, f, u, c, Y, K)
  - Let $S_K = \{(T, f, u, c, Y, K') : K' \leq K\}$ be the class of problems defined by constant K

Stereotype Set

- Principal does not know (or want to use) T and f, the true type space and distribution of types, but instead has some $S \subset T$ and $f_S \in \Delta(S)$ with $supp(f_s) = S$ and $|S| < \infty$
- (S, f_S) is the principal's approximate type space
- The principal's expected profit given menu M and assuming (S, f_s) is true is:

$$\Pi(S, M) = \sum_{t \in S} f_s(t)\left(p(y(t)) - c(y(t))\right),$$

  where $(y(t), p(y(t))) \in M$ is the item selected by an agent of type t, i.e. for all $t \in S$

$$u(t, y(t)) - p(y(t)) \geq u(t, y) - p(y) \text{ for all } (y, p(y)) \in M.$$

- The principal's true expected profit from menu M is thus $\Pi(T, M)$
- In order to take limits we need a measure of the quality of the model, ε_{true}
- Let an approximation partition $\mathcal{P}$ be a $|S|$-partition of T such that for each $A_s \in \mathcal{P}$, there is exactly one $s \in S$ such that $s \in A_s$ and so that $f_S(s) = f(A_s)$
  - Note that $A_s \in \mathcal{P}$ need not be connected
- Let $\Gamma(S, f_S)$ be the set of such partitions and note that $\Gamma(S, f_S) \neq \emptyset$
- For approximation partition $\mathcal{P}$, let

$$d(\mathcal{P}) = \sup_{A_s \in \mathcal{P}} \left\{ \sup_{t \in A_s} |s - t| \right\}$$

- The true approximation index of $(T, f; S, f_S)$ is $\varepsilon_{true} = \inf_{\mathcal{P} \in \Gamma(S, f_S)} d(\mathcal{P})$
- Let the **best approximation partition** be the one that attains ε_{true}
- Principal knows that (S, f_S) is approximate and that a true type space (T, f) exists
- Principal also knows ε such that $\varepsilon_{true} \leq \varepsilon$, for the true type space

Profit-Participation Mechanism

- For any menu $M = \left\{(y', p(y')), ..., \left(y^k, p\left(y^k\right)\right)\right\}$, the the profit-participation pricing menu be $\widetilde{M} = \left\{(y', \widetilde{p}(y')), ..., \left(y^k, \widetilde{p}\left(y^k\right)\right)\right\}$ where

$$\widetilde{p}(y) = p(y) - \tau(p(y) - c(y))$$

  such that $\tau = \sqrt{2K\varepsilon}$
- Note that in particular

$$\widetilde{p}(y) - c(y) = (1 - \tau)(p(y) - c(y))$$

- For any partition of T, $\widetilde{\mathcal{P}}$, let

$$\mathcal{S}\left(\widetilde{\mathcal{P}}, \varepsilon\right) = \left\{ S : \forall A \in \widetilde{\mathcal{P}}, \exists! s \in S \text{ such that } s \in A \right\}$$

Lemma (1)**.** *Fix T, f, S, f_S, K and $c\colon Y \to \mathbb{R}$, the cost function. Fix an approximation partition $\mathcal{P}$ and associated approximation index ε. For any feasible menu M, let $\widetilde{M}$ be the induced profit-participation pricing menu. For any $\mathcal{P}'$, a partition at least as fine as $\mathcal{P}$, and any $S' \in \mathcal{S}(\mathcal{P}', \varepsilon)$ we have:*

$$\Pi\left(S', \widetilde{M}\right) \geq \Pi(S, M) - 2\sqrt{2k\varepsilon}.$$

Proof. Fix $t \in S'$. Take $\widehat{t} \in S$, such that $\widehat{t}, t \in A \in \mathcal{P}$; hence $\left\|\widehat{t} - t\right\| \leq \varepsilon$. Let the menu option chosen by $\widehat{t} \in S$ be $(\widehat{y}, p(\widehat{y})) \in M$. Either $t \in S'$ chooses good $\widehat{y}$ (case 1) or not (case 2).

If t chooses $(\widehat{y}, \widetilde{p}(\widehat{y})) \in \widetilde{M}$ then by definition:

$$\widetilde{p}(\widehat{y}) - c(\widehat{y}) = (1 - \tau)(p(\widehat{y}) - c(\widehat{y})).$$

Agent $\widehat{t}$ always has the outside option, so that:

$$p(\widehat{y}) - c(\widehat{y}) \leq u\left(\widehat{t}, \widehat{y}\right) - c(\widehat{y}) \leq \Pi_{\max} = 1.$$

Therefore in case 1, the loss is bounded by

$$\widetilde{p}(\widehat{y}) - c(\widehat{y}) - [p(\widehat{y}) - c(\widehat{y})] \geq -\sqrt{2K\varepsilon}. \tag{1}$$

Consider case 2, where t chooses $(y', \widetilde{p}(y')) \in \widetilde{M}$ where $y' \neq \widehat{y}$. Note that since $\widehat{t}$ chooses $\widehat{y}$ over y' from M

$$u\left(\widehat{t}, \widehat{y}\right) - p(\widehat{y}) \geq u\left(\widehat{t}, y'\right) - p(y').$$

Since t' chooses y' over $\widehat{y}$ from $\widetilde{M}$:

$$u(t, \widehat{y}) - \widetilde{p}(\widehat{y}) \leq u(t, y') - \widetilde{p}(y').$$

Subtracting the latter from the former yields:

$$\begin{aligned} & u(t, \widehat{y}) - u(t, y') - \left(u\left(\widehat{t}, \widehat{y}\right) - u\left(\widehat{t}, y'\right)\right) \\ \leq\ & p(y') - \widetilde{p}(y') - (p(\widehat{y}) - \widetilde{p}(\widehat{y})) \\ =\ & \tau(p(y') - c(y')) - \tau(p(\widehat{y}) - c(\widehat{y})). \end{aligned} \tag{2}$$

To bound inequality 2, note that Lipschitz continuity of u implies:

$$\left|\frac{u\left(\widehat{t}, \widehat{y}\right) - u(t, \widehat{y})}{\left\|\widehat{t} - t\right\|}\right| \leq k,$$

and since $\left\|\widehat{t} - t\right\| \leq \varepsilon$, we have that $\left|u\left(\widehat{t}, \widehat{y}\right) - u(t, \widehat{y})\right| \leq k\varepsilon$. Similarly, $\left|u\left(\widehat{t}, y'\right) - u(t, y')\right| \leq k\varepsilon$, so that

$$u(t, \widehat{y}) - u(t, y') - \left(u\left(\widehat{t}, \widehat{y}\right) - u\left(\widehat{t}, y'\right)\right) \geq -2k\varepsilon$$

and thus

$$p(y') - c(y') - (p(\widehat{y}) - c(\widehat{y})) \geq \frac{-2k\varepsilon}{\tau}.$$

The loss in case 2 is $\widetilde{p}(y') - c(y') - [p(\widehat{y}) - c(\widehat{y})]$, or

$$\begin{aligned} & \widetilde{p}(y') - c(y') - [p(y') - c(y')] + p(y') - c(y') - [p(\widehat{y}) - c(\widehat{y})] \\ & \quad \geq -\tau[p(y') - c(y')] - \frac{2k\varepsilon}{\tau} \geq -\tau - \frac{2k\varepsilon}{\tau}. \end{aligned}$$

Choosing τ to minimize $\tau + \frac{2k\varepsilon}{\tau}$, yields $\tau = \sqrt{2k\varepsilon}$ and thus

$$\widetilde{p}(y') - c(y') - [p(\widehat{y}) - c(\widehat{y})] \geq -2\sqrt{2k\varepsilon}. \tag{3}$$

Given inequalities 1 and 3, we take expecations to get the desired result. □

Profit-Participation Mechanism

- The profit-participation mechanism (PPM) is as follows:
  1. Given S, f_S, Y, u, c, and ε find the optimal menu M
  2. Apply profit-participation pricing to M and offer resulting menu $\widetilde{M}$
- The PPM loss is defined as

$$\Pi(T, M^*) - \Pi\left(T, \widetilde{M}\right)$$

where M^* is the optimal menu if the principal knew (T, f)

Theorem (1). *The PPM loss is bounded above by $4\sqrt{2k\varepsilon}$.*

Proof. Let $M^* = \underset{M=\{(y(t),p(t))\}}{\arg\max} \Pi(T, M)$ subject to

$$u(t, y(t)) - p(t) \geq u(t, y(t')) - p(t') \quad \text{for all } t, t' \in T$$

Let $\mathcal{P}$ be the best approximation partition for $(T, f; S, f_S)$ and let

$$S_{\max} = \underset{S \in \mathcal{S}(\mathcal{P},\varepsilon)}{\arg\max} \Pi(S, M^*).$$

Note that $\Pi(S_{\max}, M^*) \geq \Pi(T, M^*)$. Let M' be the profit-participation pricing menu derived from M^*. Fix any $S \in \mathcal{S}(\mathcal{P}, \varepsilon)$ and note that lemma 1 implies:

$$\Pi(S, M') \geq \Pi(S_{\max}, M^*) - 2\sqrt{2k\varepsilon}.$$

Let $\widehat{M} = \arg\max_M \Pi(S, M)$ subject to usual IC constraints, so that $\Pi\left(S, \widehat{M}\right) \geq \Pi(S, M')$. Let $\widetilde{M}$ be the profit-participation pricing menu derived from $\widehat{M}$; using lemma 1 again we have that $\Pi\left(T, \widetilde{M}\right) \geq \Pi\left(S, \widehat{M}\right) - 2\sqrt{2k\varepsilon}$. Putting the above together gives the desired result:

$$\begin{aligned} \Pi\left(T, \widetilde{M}\right) & \geq \Pi\left(S, \widehat{M}\right) - 2\sqrt{2k\varepsilon} \geq \Pi(S, M') - 2\sqrt{2k\varepsilon} \\ & \geq \Pi(S_{\max}, M^*) - 4\sqrt{2k\varepsilon} \geq \Pi(T, M^*) - 4\sqrt{2k\varepsilon}. \end{aligned}$$

□

Alternative Mechanisms

- No-Free-Lunch theorem
- A mechanism is model-based if it can be represented as a two-step process where the price found in step 1 of PPM is modified by some function so that the modified prices are

$$\widetilde{p}(y) = \Psi(p(y), c(y), k, \varepsilon)$$

- A model-based mechanism violates profit participation if for some $\varepsilon_{true} > 0$, there exists $\overline{p} > 0$ and $\overline{c} > 0$ such that for all $p' < p'' \leq \overline{p}$ and $c \leq \overline{c}$

$$p'' - \Psi(p'', c, k, \varepsilon) \leq p' - \Psi(p', c, k, \varepsilon)$$

- PPM is model-based and does not violate profit participation

Theorem (2). *For any $k > 0$, the upper bound to the profit loss generated by a model-based mechanism that violates profit participation does not vanish as $\varepsilon \to 0$.*

- If a mechanism is to achieve efficient profit in the limit, then either it is similar to PPM or very different
- Intuition is that non-profit-participation model-based mechanisms have many binding constraints
- Better to add slack to constraints so that agents choose alternatives which generate a higher profit

Proof. Consider a model-based mechanism given by Ψ that violates profit participation for $\overline{p} > 0$ and $\overline{c} > 0$. Let $p' = \frac{1}{2}\overline{p}$, $p'' \in \left(\frac{1}{2}\overline{p}, \min\left\{\frac{1}{2}\overline{p} + \frac{1}{6}k, \overline{p}\right\}\right)$ and $c = 0$, so that:

$$p'' - \Psi(p'', 0, k, \varepsilon) \leq p' - \Psi(p', 0, k, \varepsilon).$$

Consider a screening problem as follows: $T = [0,2]$, $f(t) = \frac{1}{2}$ for all $t \in T$, $Y = [1,2] \cup \{\overline{y}, y_0\}$ and

$$u(t,y) = \begin{cases} p' + q(t - 1 - 2|y - t|) & \text{if } y \in [1,2] \\ p' & \text{if } y = \overline{y} \\ 0 & \text{if } y = y_0 \end{cases},$$

where $p'' - p' = q$.

Optimal for types $t \leq 1$ to buy $\overline{y}$ at price p'; for types $t > 1$ to get $\widehat{y}(t) = t$ at price $p' + q(t-1)$. Principal's expected profit is $p' + \frac{1}{4}q$. This problem satisfies the k-Lipschitz condition since:

$$\max_{(t,y)} \left| \frac{\partial u(t,y)}{\partial t} \right| \leq 3q = 3p'' - 3p' < \frac{1}{2}k.$$

Now, take the following sequence of approximate type spaces:

$$S_n = \left\{ \frac{j}{2^n} : j \in \mathbb{N}, j \leq 2^{n+1} \right\},$$

$$f_{S_n}(0) = f_{S_n}(2) = \frac{1}{2^{n+2}}, \text{ and } f_{S_n}(s) = \frac{1}{2^{n+1}} \text{ for other } s.$$

Note that $\varepsilon_{true}^n = \frac{1}{2}\frac{1}{2^n} = 2^{-n-1}$ and so we can set $\varepsilon_n = 2^{-n-1}$. Fix n and write $\Psi(\cdot,\cdot) = \Psi(\cdot,\cdot,K,\varepsilon_n)$. Note that since $p(\overline{y}) = p'$ and $p(\widehat{y}(s)) = p' + q(s-1)$, we have that the mechanism Ψ gives prices:

$$\widetilde{p}(\overline{y}) = \Psi(p', 0) \text{ and } \widetilde{p}(y) = \Psi(p' + q(s-1), 0).$$

Since Ψ violates profit participation, for $t \in [1,2]$:

$$p' + q(s-1) - \Psi(p' + q(s-1), 0) \leq p' - \Psi(p', 0). \tag{4}$$

Thus, any $t \in [1,2] \smallsetminus S_n$ gets utility:

$$\begin{matrix} p' + q(t - 1 - 2|s-t|) - \Psi(p' + q(s-1), 0) & \text{if } t \text{ gets } \widehat{y}(s) \\ p' - \Psi(p', 0) & \text{if } t \text{ gets } \overline{y} \end{matrix}.$$

Hence $t \in [1,2] \smallsetminus S_n$ would choose $\widehat{y}(s)$ if $q(t - 1 - 2|s-t|) - \Psi(p' + q(s-1), 0) \geq -\Psi(p', 0)$. Together with equation 4 this implies:

$$\begin{aligned} q(t - 1 - 2|s-t|) - q(s-1) &\geq 0, \text{ or} \\ t - s &\geq 2|s-t|, \end{aligned}$$

which is a contradiction.

Therefore a measure 1 of agents chooses $\overline{y}$ instead of a personalized alternative. Hence the principal's profit for any S_n is p'. Note that as $n \to \infty$ and $\varepsilon_n = \frac{1}{2^{n+1}} \to 0$, but the principals profit converges to $p' < p' + \frac{q}{2}$. □

Comments on Proofs

- The proof of Lemma 1 contains an incorrect claim in the second paragraph (although the validity of the lemma remains intact)
  - The argument for why the required $\widehat{t} \in S$ exists and why $\|t - \widehat{t}\| \leq \varepsilon$ needs to be revisited
- The claim in the second paragraph, that $S' \subseteq S$, is false in general, since $|S'| > |S|$ unless $\mathcal{P}' = \mathcal{P}$
- Furthermore, if the statement was amended to read $S \subseteq S'$, there would still be a counterexample

Example. *Take $\mathcal{P}$ to be a regular square grid where the individual squares have length l and take $\varepsilon = \frac{1}{2}l$. In this case we have that $\mathcal{S}(\mathcal{P},\varepsilon)$ is a singleton and in particular $\mathcal{S}(\mathcal{P},\varepsilon) = \{S\}$ where S is the set of all types which are at the centre of each square. Let $\mathcal{P}'$ be a partition finer than this (see diagram). Note that $\mathcal{S}(\mathcal{P}',\varepsilon)$ is not a singleton and that one can easily find an $S' \in \mathcal{S}(\mathcal{P}',\varepsilon)$, such that $S \subseteq S'$.*

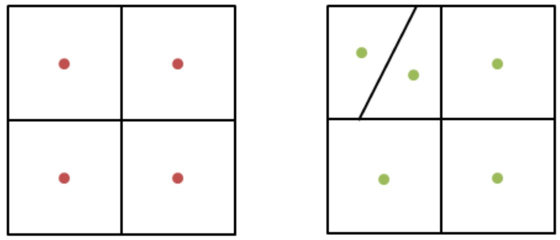

Figure 1: The square grid $\mathcal{P}$ and S, the set of red dots is shown on the left. The right pane shows $\mathcal{P}'$, a partition finer than $\mathcal{P}$, and S', the set of red dots. Note that $S \not\subseteq S'$.

- One could correctly make the claim that for any $S \in \mathcal{S}(\mathcal{P},\varepsilon)$ there exists a $S' \in \mathcal{S}(\mathcal{P}',\varepsilon)$ such that $S \subseteq S'$, but this is not helpful for the proposition
- That one can find a $\widehat{t} \in S$ such that $\|t - \widehat{t}\| \leq \varepsilon$ follows since for any $t \in T$, there exists some $A \in \mathcal{P}$, such that $t \in A$. Further there exists some $\widehat{t} \in S$ such that $\widehat{t} \in A$ and since

$$\varepsilon \geq \varepsilon_{true} = \inf_{\mathcal{P} \in \Gamma(S,f_S)} \left(\sup_{A_s \in \mathcal{P}} \left\{ \sup_{t \in A_s} |s - t| \right\} \right),$$

  it follows that $\|t - \widehat{t}\| \leq \varepsilon$

Remark (Dec 2012). *It appears that the proof of the lemma has been corrected in the latest version of the paper.*